J Stat Phys (2009) 137: 917–935
DOI 10.1007/s10955-009-9831-0

Superdiffusivity of the 1D Lattice Kardar-Parisi-Zhang Equation

Tomohiro Sasamoto · Herbert Spohn

Received: 24 August 2009 / Accepted: 16 September 2009 / Published online: 7 October 2009

Abstract The continuum Kardar-Parisi-Zhang equation in one dimension is lattice discretized in such a way that the drift part is divergence free. This allows to determine explicitly the stationary measures. We map the lattice KPZ equation to a bosonic field theory which has a cubic anti-hermitian nonlinearity. Thereby it is established that the stationary two-point function spreads superdiffusively.

Keywords Continuum growth model · Non-hermitian bosonic field theory

1 Introduction

Oversimplified models of surface growth have remained of interest, in particular their one-dimensional version. In very general terms, one investigates a conserved field with a stochastic dynamics which does not satisfy the condition of detailed balance. Mostly models with discrete space and discrete heights have been in focus, as e.g. the asymmetric exclusion processes (ASEP), also its totally asymmetric version (TASEP), and the polynuclear growth (PNG) model. We refer to [1–3] for reviews. Some features of these models are related to integrable systems and random matrix theory, which has become an independent motivation for intense study. There are unexpected connections to random tilings, as the Aztec diamond, and to crystal shapes, somewhat less surprising to directed last passage percolation, also known as directed polymer in a random medium. For several features of ASEP, TASEP, and PNG exact solutions can be obtained through a rather intricate asymptotic analysis. In particular, the stationary two-point function has been computed in the scaling limit [4, 5]. Except for model dependent scale factors one finds the same scaling function in the various

On the occasion of the 100th Statistical Mechanics Meeting, December 2008 at Rutgers University, we dedicate this article to Joel Lebowitz as mentor and friend for so many years.

T. Sasamoto · H. Spohn (✉)
Zentrum Mathematik, TU München, 85747 Garching, Germany
e-mail: spohn@ma.tum.de

T. Sasamoto
e-mail: sasamoto@ma.tum.de

models, supporting, at least partially, the universality hypothesis for growth models with local deposition rules and neglected surface diffusion.

In a seminal paper, Kardar, Parisi, and Zhang [6] proposed a continuum equation for surface growth. In one dimension their equation reads

$$\partial_t h(x,t) = \frac{1}{2}\lambda_{\mathrm{b}}(\partial_x h(x,t))^2 + \nu_{\mathrm{b}}\partial_x^2 h(x,t) + \sqrt{D_{\mathrm{b}}}\xi(x,t). \tag{1.1}$$

Here h is the height function over $\mathbb{R}$ at time t, λ_{b} is the strength of the nonlinear growth velocity, ν_{b} is the strength of the local smoothening, and ξ is normalized Gaussian space-time white noise, $\langle \xi(x,t)\xi(x',t')\rangle = \delta(x-x')\delta(t-t')$. The slope $u = \partial_x h$ then satisfies the stochastic Burgers equation,

$$\partial_t u = \frac{1}{2}\lambda_{\mathrm{b}}\partial_x u^2 + \nu_{\mathrm{b}}\partial_x^2 u + \sqrt{D_{\mathrm{b}}}\partial_x \xi. \tag{1.2}$$

Note that this relation is a special property of one dimension.

Equation (1.2) should be viewed as a conservation law, where the current is the sum of a nonlinear systematic piece, a diffusive term proportional to the gradient, and a stochastic component. Therefore it is expected that (1.2) is in the same universality class as ASEP and PNG. The most convincing evidence comes from the numerical simulation of the discretized version of (1.1) with 1024 lattice sites [7]. The stationary two-point function is computed and good agreement is obtained with the exact solution from TASEP and PNG over a substantial range of wave vectors.

On the theoretical side, one notes that the noise in (1.2) is very singular. To cope with this difficulty, one worked out approach [8] is to make (1.1) linear through the Cole-Hopf transformation

$$Z(x,t) = \exp[(\lambda_{\mathrm{b}}/2\nu_{\mathrm{b}})h(x,t)]. \tag{1.3}$$

Then the "partition function" $Z(x,t)$ satisfies

$$\partial_t Z(x,t) = \nu_{\mathrm{b}}\partial_x^2 Z(x,t) + \frac{\lambda_{\mathrm{b}}\sqrt{D_{\mathrm{b}}}}{2\nu_{\mathrm{b}}}\xi(x,t)Z(x,t), \tag{1.4}$$

with initial conditions $Z(x,0) > 0$. Being linear, one can give sense to $Z(x,t)$ as a stochastic process with continuous sample paths. In particular, $Z(x,t) > 0$ with probability 1 and one *defines* $h(x,t)$ through (1.3). In fact, this type of solution of the KPZ equation can be recovered by a suitable continuum limit of the ASEP with a properly chosen weak asymmetry.

An alternative approach is to regularize the noisy Burgers equation (1.2). Depending on the point of view there are then two choices. The fluid dynamics camp regards (1.2) as oversimplified large scale randomly stirred Navier-Stokes equations and studies small scale properties of the solution, see [12, 13] out of a large body of literature. But then it is natural to keep the δ-function in time and to replace the δ-function in space by a smoothened version. On the other hand the surface growth community knows that the lattice structure of the solid defines a smallest length scale. Thus one discretizes the KPZ equation, as done without further ado in any numerical simulation, and studies its properties on large scales. In relation to ASEP and PNG it is natural to follow the second route and to discretize $\mathbb{R}$ as $\delta\mathbb{Z}$ with lattice constant $\delta > 0$. The field variables are denoted by $u_j(t) \in \mathbb{R}$, $j \in \mathbb{Z}$, as the lattice version of $u(x,t)$ in (1.2). Then the lattice KPZ equation, studied in this paper, reads

$$\frac{\mathrm{d}}{\mathrm{d}t}u_j = \frac{1}{2}(\lambda_\mathrm{b}/3\delta)(u_{j+1}^2 + u_j u_{j+1} - u_{j-1}u_j - u_{j-1}^2) + (\nu_\mathrm{b}/\delta^2)(u_{j+1} - 2u_j + u_{j-1}) + (D_\mathrm{b}/\delta)^{1/2}(\xi_j - \xi_{j-1}), \quad j \in \mathbb{Z}. \tag{1.5}$$

Here $\{\xi_j, j \in \mathbb{Z}\}$ is a collection of independent and normalized white noises. Below we will determine a family of stationary measures of (1.5). We consider then the space-time stationary solutions to (1.5) and the object of prime interest will be stationary two-point function

$$S_\rho(j,t) = \langle u_j(t)u_0(0)\rangle_\rho - \langle u_0(0)\rangle_\rho^2 \tag{1.6}$$

at average slope $\rho = \langle u_j(t)\rangle_\rho$.

Each one of the three terms on the right hand side of (1.5) converges as $\delta \to 0$ to the corresponding term in (1.2). To obtain the scale invariant theory, however, one has to adjust the "coupling constants" in (1.5) in such a way that $S(j,t)$ converges to a nontrivial limit. This is the reason for the subscript b, which stands for bare coupling parameters.

To provide a brief summary, in Sect. 2 we rewrite (1.5) as a bosonic field theory with a cubic anti-hermitian nonlinearity, where for simplicity we consider only the case of zero slope, $\rho = 0$. As explained in Sect. 3, restricting the bosonic occupation variables to 0,1, i.e. restricting to hard core bosons, yields precisely the stationary ASEP at density $1/2$. The bosonic representation is used in the study of the spreading of $S_{\rho=0}(j,t)$. $S_{\rho=0}(j,t)$ is even in j and the normalized variance reads

$$\mathrm{Var}\,(t) = \chi^{-1}\sum_{j\in\mathbb{Z}} j^2 S_{\rho=0}(j,t), \quad \chi = \sum_{j\in\mathbb{Z}} S_{\rho=0}(j,0). \tag{1.7}$$

In Sect. 5 we will establish the upper and lower bounds as

$$t^{5/4} \le \mathrm{Var}\,(t) \le t^{3/2} \tag{1.8}$$

as $t \to \infty$. We then develop an iterative scheme based on the relaxation time approximation and recover the KPZ prediction of $\mathrm{Var}\,(t) \cong t^{4/3}$. Very recently J. Quastel announced a proof of the following bounds,

$$c_- t^{4/3} \le \mathrm{Var}_{\mathrm{CH}}(t) \le c_+ t^{4/3} \tag{1.9}$$

for large t and suitable constants $0 < c_- < c_+$ [9]. Here $\mathrm{Var}_{\mathrm{CH}}(t)$ is computed from the variance of $\log Z(x,t)$ with $Z(x,t)$ the Cole-Hopf solution of (1.4) with initial data $Z(x,0)$ such that $\log Z(x,0)$ is two-sided Brownian motion in x. In the final section the continuum approximation is discussed and contrasted with the well studied case of (1.2) for $D_\mathrm{b} = 0$, $\nu_\mathrm{b} \to 0$, and white noise initial data.

After completion of our manuscript we learned about a paper by C. Bernardin, where similar techniques are used in the study of a driven energy exchange model [10].

2 The Lattice KPZ Equation

To simplify notation we set $\lambda_0 = \lambda_\mathrm{b}/\delta$, $\nu_0 = \nu_\mathrm{b}/\delta^2$, $D_0 = D_\mathrm{b}/\delta$, and $\alpha = \nu_0/D_0$.

The lattice KPZ equation (1.5) conserves the slope field u, which becomes manifest through introducing the current function

$$
\begin{aligned}
w_j &= \frac{1}{6}\lambda_0(u_j^2 + u_j u_{j+1} + u_{j+1}^2) + \nu_0(u_{j+1} - u_j) \\
&= \tilde{w}_j + \nu_0(u_{j+1} - u_j).
\end{aligned}
\tag{2.1}
$$

Then

$$
\frac{d}{dt}u_j = w_j - w_{j-1} + \sqrt{D_0}(\xi_j - \xi_{j-1}), \quad j \in \mathbb{Z}. \tag{2.2}
$$

In principle, there are many possibilities to discretize $\partial_x u(x)^2$ in (1.2). Our choice is singled out by the facts (i) the discretization involves only nearest neighbors and (ii) the drift term is a divergence free vector field, i.e.

$$
\sum_{j\in\mathbb{Z}} \partial_j(w_j - w_{j-1}) = 0, \tag{2.3}
$$

where $\partial_j = \partial/\partial u_j$. For this particular discretization one can compute explicitly the invariant measures. Equations (2.1) and (2.2) was first proposed in [14], see also [15] for a more recent study.

For a while we study (2.2) on the ring $[1, \ldots, N]$, i.e.

$$
u_{N+j} = u_j \quad \text{and} \quad \xi_{N+j} = \xi_j. \tag{2.4}
$$

Since $\sum_{j=1}^N u_j$ is conserved, the dynamics is well defined. On the other hand, the infinite lattice dynamics would require a separate study. Eventually we will consider only the stationary case, for which the dynamics can be constructed using semi-group theory [11]. The generator for the diffusion process (2.2) is then

$$
L_N = \sum_{j=1}^N \left[(w_j - w_{j-1})\partial_j + \frac{1}{2}D_0(\partial_j - \partial_{j-1})^2 \right]. \tag{2.5}
$$

Noting the identity

$$
\sum_{j=1}^N (w_j - w_{j-1})u_j = 0, \tag{2.6}
$$

one verifies that, for every $\rho \in \mathbb{R}$, the independent Gaussians

$$
\prod_{j=1}^N \left\{ (\alpha/\pi)^{1/2} \exp[-\alpha(u_j - \rho)^2] \right\} = \left(\psi_{0,\rho}^N(\underline{u}) \right)^2 \tag{2.7}
$$

are invariant for (2.5), i.e. for every smooth function f on configuration space it holds

$$
\int_{\mathbb{R}^N} \mathrm{d}\underline{u}\, \psi_{0,\rho}^N(\underline{u})^2 L_N f(\underline{u}) = 0. \tag{2.8}
$$

Note that the stationary measure does not depend on λ_0. Let us denote the average with respect to $(\psi_{0,\rho}^N)^2$ by $\langle\cdot\rangle_{\rho,N}$, $\langle 1\rangle_{\rho,N} = 1$. Then

$$
\langle u_j \rangle_{\rho,N} = \rho \tag{2.9}
$$

is the average slope,

$$\langle u_j^2 \rangle_{\rho,N} - \langle u_j \rangle_{\rho,N}^2 = \frac{1}{2\alpha} \tag{2.10}$$

the variance, and

$$\langle w_j \rangle_{\rho,N} = \frac{1}{6}\lambda_0\big(\alpha^{-1} + 3\rho^2\big) = j(\rho) \tag{2.11}$$

the average current. For simplicity we will consider only the slope 0 case, $\rho = 0$, and hence omit the index ρ.

We switch from the generator L_N to the Hamiltonian H_N through the ground state transformation

$$\psi_0^N L_N (\psi_0^N)^{-1} = -H_N. \tag{2.12}$$

Then

$$H_N = \frac{1}{2} D_0 \sum_{j=1}^{N} \big(-(\partial_j - \partial_{j-1})^2 + \alpha^2 (u_j - u_{j-1})^2 - 2\alpha \big) - \sum_{j=1}^{N} (\tilde{w}_j - \tilde{w}_{j-1}) \partial_j. \tag{2.13}$$

We introduce the standard annihilation/creation operators at site j through

$$a_j = \frac{1}{\sqrt{2\alpha}}(\alpha u_j + \partial_j), \qquad a_j^* = \frac{1}{\sqrt{2\alpha}}(\alpha u_j - \partial_j). \tag{2.14}$$

Note that they satisfy the canonical commutation relations

$$[a_i, a_j^*] = \delta_{ij}. \tag{2.15}$$

Then

$$H_N = \tilde{H}_{0,N} + \lambda_0 (\tilde{A}_N^* - \tilde{A}_N), \tag{2.16}$$

where

$$\tilde{H}_{0,N} = \nu_0 \sum_{j=1}^{N} (a_{j+1} - a_j)^* (a_{j+1} - a_j) \tag{2.17}$$

and

$$\tilde{A}_N = (3 \cdot 2^{3/2})^{-1} \alpha^{-1/2} \sum_{j=1}^{N} \big(a_j a_{j+1}^* a_{j+1} + a_j a_j a_{j+1}^* - a_j^* a_{j+1} a_{j+1} - a_j^* a_j a_{j+1} \big). \tag{2.18}$$

We want to compute the propagator $\exp[-tH_N]$, $t \geq 0$. To reduce the number of parameters we rescale time as $t \rightsquigarrow \nu_0 t$. Then the prefactor of $H_{0,N}$ becomes one. We also introduce the coupling constant

$$\lambda = (3 \cdot 2^{3/2})^{-1} \lambda_0 \alpha^{-1/2} \nu_0^{-1}. \tag{2.19}$$

Then

$$H_N = H_{0,N} + \lambda (A_N^* - A_N), \tag{2.20}$$

where

$$H_{0,N} = \sum_{j=1}^{N} (a_{j+1} - a_j)^*(a_{j+1} - a_j), \tag{2.21}$$

$$A_N = \sum_{j=1}^{N} \big(a_j a_{j+1}^* a_{j+1} + a_j a_j a_{j+1}^* - a_j^* a_{j+1} a_{j+1} - a_j^* a_j a_{j+1}\big). \tag{2.22}$$

It is convenient to use the *Fock space* representation of the bosonic field a_j, a_j^*. Physically this corresponds to a study of local excitations away from the Gaussian stationary measure, as e.g. encoded by the two-point function $S(j,t)$. For $N = \infty$ the Fock space $\mathfrak{F}$ is over $\ell_2 = \ell_2(\mathbb{Z})$ with the n-particle space $\mathfrak{F}_n = (\ell_2)_{\mathrm{sym}}^{\otimes n}$ and

$$\mathfrak{F} = \bigoplus_{n=0}^{\infty} \mathfrak{F}_n. \tag{2.23}$$

An element $f \in \mathfrak{F}$ is a sequence $\{f_0, f_1, \dots, f_n, \dots\}$. $f_n : \mathbb{Z}^n \to \mathbb{C}$ and f_n is symmetric in its arguments, $f_0 \in \mathbb{C}$. The scalar product on $\mathfrak{F}_n$ is

$$\langle f, g\rangle_n = \sum_{(x_1,\dots,x_n)\in\mathbb{Z}^n} f_n(x_1,\dots,x_n)^* g_n(x_1,\dots,x_n). \tag{2.24}$$

$f \in \mathfrak{F}$ if and only if

$$\langle f, f\rangle = |f_0|^2 + \sum_{n=1}^{\infty} \langle f_n, f_n\rangle_n < \infty. \tag{2.25}$$

$\mathfrak{F}_n$ is the n-particle subspace. The Fock vacuum is the vector $\Omega = (1, 0, 0, \dots)$.

The $\{a_j, a_j^*, j \in \mathbb{Z}\}$ are the usual bosonic annihilation/creation operators on $\mathfrak{F}$, which are defined through

$$(a_j f_{n+1})(x_1,\dots,x_n) = \sqrt{n+1}\, f_{n+1}(j, x_1,\dots,x_n) \tag{2.26}$$

and

$$(a_j^* f_{n-1})(x_1,\dots,x_n) = \frac{1}{\sqrt{n}} \sum_{\ell=1}^{n} \delta(x_\ell - j) f_{n-1}(x_1,\dots,\hat{x}_\ell,\dots,x_n) \tag{2.27}$$

with the convention that $\hat{x}_\ell$ means the omission of x_ℓ from the configuration $(x_1, \dots, x_n)$, see e.g. [16]. From (2.20)–(2.22) in the limit $N \to \infty$, we arrive at

$$H = H_0 + \lambda(A^* - A). \tag{2.28}$$

Explicitly

$$H_0 = \sum_{j\in\mathbb{Z}} (a_{j+1} - a_j)^*(a_{j+1} - a_j) \tag{2.29}$$

and

$$A = \sum_{j\in\mathbb{Z}} \big(a_j a_{j+1}^* a_{j+1} + a_j a_j a_{j+1}^* - a_j^* a_{j+1} a_{j+1} - a_j^* a_j a_{j+1}\big) \tag{2.30}$$

as operators on Fock space.

$H_0 \restriction \mathfrak{F}_n$ is the Laplacian and corresponds to n independent random walks on $\mathbb{Z}$ with nearest neighbor hopping of rate 1. Clearly, $H_0 = H_0^*$, while $A^* - A$ is antisymmetric, $(A^* - A)^* = -(A^* - A)$. Using (2.26), (2.27), the action of A on Fock space is represented by

$$\begin{aligned} Af_{n+1}(x_1, \dots, x_n) = \sqrt{n+1} \sum_{\ell=1}^{n} \big(& f_{n+1}(x_1, \dots, x_n, x_\ell - 1) - f_{n+1}(x_1, \dots, x_n, x_\ell + 1) \\ & + f_{n+1}(x_1, \dots, \hat{x}_\ell, \dots, x_n, x_\ell - 1, x_\ell - 1) \\ & - f_{n+1}(x_1, \dots, \hat{x}_\ell, \dots, x_n, x_\ell + 1, x_\ell + 1)\big). \end{aligned} \tag{2.31}$$

To have an example, let us consider the two-point function at slope zero,

$$S(j,t) = \mathbb{E}(u_j(t)u_0(0)). \tag{2.32}$$

Here $N = \infty$ and the expection $\mathbb{E}(\cdot)$ refers to the solution of (2.1), (2.2) with $u_j(0)$ distributed as independent Gaussians with mean 0 and variance $1/2\alpha$. In Fock space $S(j,t)$ is expressed through

$$S(j,t) = \frac{1}{2\alpha} \langle \Omega, a_0 \mathrm{e}^{-\nu_0 t H} a_j^* \Omega \rangle, \quad t \geq 0. \tag{2.33}$$

Note that by the reflection $u_j \rightsquigarrow -u_{-j}$ and by time stationarity, one concludes

$$S(j,t) = S(-j,t), \qquad S(j,t) = S(j,-t), \quad t \geq 0. \tag{2.34}$$

Computationally it is often more convenient to switch to Fourier space. For $f : \mathbb{Z} \to \mathbb{C}$ we define the Fourier transform

$$\widehat{f}(k) = \sum_{j \in \mathbb{Z}} \mathrm{e}^{-i2\pi kj} f(j) \tag{2.35}$$

with inverse

$$f(j) = \int_0^1 \mathrm{d}k \mathrm{e}^{i2\pi kj} \widehat{f}(k). \tag{2.36}$$

Thus the one-particle space is now $L^2([0,1], \mathrm{d}k)$. We set $\mathbb{T} = [0,1]$ as first Brillouin zone. Correspondingly

$$a(k) = \sum_{j \in \mathbb{Z}} \mathrm{e}^{-i2\pi kj} a_j. \tag{2.37}$$

Note that

$$[a(k), a(k')^*] = \delta(k - k'), \tag{2.38}$$

which means that the corresponding map between Fock spaces is unitary. In Fourier representation it holds

$$H_0 = \int_{\mathbb{T}} \mathrm{d}k \omega(k) a(k)^* a(k), \quad \omega(k) = 2(1 - \cos(2\pi k)) \tag{2.39}$$

and

$$\begin{aligned} A = -i \int_{\mathbb{T}^3} & \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3 \delta(k_1 - k_2 - k_3)\big(2\sin(2\pi k_1) \\ & + \sin(2\pi k_2) + \sin(2\pi k_3)\big) a(k_1)^* a(k_2) a(k_3). \end{aligned} \tag{2.40}$$

3 Relation to the ASEP at Density 1/2

The partially asymmetric simple exclusion process (ASEP) is a stochastic particle system on $\mathbb{Z}$ with hard exclusion, i.e. the occupation variable η_j at site j takes only the values 0,1. The state space is $\{0,1\}^{\mathbb{Z}}$. Particles hop to the right neighbor with rate $1+p$ and to the left neighbor with rate $1-p$, $|p|\le 1$, provided the destination site is empty. Therefore the generator, L_{AS}, reads

$$L_{\mathrm{AS}} f(\eta)=\sum_{j\in\mathbb{Z}}\big((1+p)\eta_j(1-\eta_{j+1})+(1-p)(1-\eta_j)\eta_{j+1}\big)\big(f(\eta^{jj+1})-f(\eta)\big), \quad (3.1)$$

where η^{jj+1} denotes the configuration η with the occupations at sites j and $j+1$ interchanged. The Bernoulli measures are invariant under L_{AS}. We consider only the stationary process with Bernoulli $1/2$.

We expand L_{AS} in the natural basis for the Bernoulli measure with density $1/2$ [17]. The basis functions are labeled by finite subsets, Λ, of $\mathbb{Z}$ and are of the form

$$\psi_\Lambda(\eta)=\prod_{j\in\Lambda}(2\eta_j-1). \quad (3.2)$$

In particular, with $\langle\cdot\rangle_{1/2}$ denoting average over Bernoulli $1/2$,

$$\langle\psi_{\Lambda_1}\psi_{\Lambda_2}\rangle_{1/2}=\delta(\Lambda_1,\Lambda_2). \quad (3.3)$$

A general function, v, is represented by

$$v(\eta)=\sum_{\Lambda\subset\mathbb{Z}}\hat{v}(\Lambda)\psi_\Lambda(\eta), \qquad \sum_{\Lambda\subset\mathbb{Z}}|\hat{v}(\Lambda)|^2<\infty, \quad (3.4)$$

the sum being over all finite subsets of $\mathbb{Z}$. In this basis L_{AS} is represented by

$$L_{\mathrm{AS}}=-H_{\mathrm{AS}}, \qquad H_{\mathrm{AS}}=\mathcal{S}+p(\mathcal{A}^*-\mathcal{A}). \quad (3.5)$$

Here $\mathcal{S}=\mathcal{S}^*$ and

$$\mathcal{S}\hat{v}(\Lambda)=-\sum_{x\in\mathbb{Z}}\big(\hat{v}(\Lambda_{x,x+1})-\hat{v}(\Lambda)\big), \quad (3.6)$$

where $\Lambda_{x,x+1}$ is obtained from Λ by exchanging the occupancies at x and $x+1$. To define $\mathcal{A}$ and $\mathcal{A}^*$ we introduce the outer left boundary of Λ, $\ell(\Lambda)=\{x\mid x\notin\Lambda, x+1\in\Lambda\}$, and the outer right boundary of Λ, $r(\Lambda)=\{x\mid x\notin\Lambda, x-1\in\Lambda\}$. Correspondingly, the inner left and inner right boundary of Λ are $\bar{\ell}(\Lambda)=\{x\mid x\in\Lambda, x-1\notin\Lambda\}$, $\bar{r}(\Lambda)=\{x\mid x\in\Lambda, x+1\notin\Lambda\}$. Then

$$\begin{aligned}\mathcal{A}\hat{v}(\Lambda)&=\sum_{x\in\ell(\Lambda)}\hat{v}(\Lambda\cup\{x\})-\sum_{x\in r(\Lambda)}\hat{v}(\Lambda\cup\{x\}),\\ \mathcal{A}^*\hat{v}(\Lambda)&=\sum_{x\in\bar{\ell}(\Lambda)}\hat{v}(\Lambda\setminus\{x\})-\sum_{x\in\bar{r}(\Lambda)}\hat{v}(\Lambda\setminus\{x\}).\end{aligned} \quad (3.7)$$

L_{AS} is the generator in the number space representation. To be able to compare with H one still has to transform unitarily to Fock space representation. For this purpose let $|\Lambda|=n$,

$\Lambda = \{x_1, \dots, x_n\}$. Out of $\hat{v}(\Lambda)$ we construct $f_n \in \mathfrak{F}_n$ by

$$f_n(x_1, \dots, x_n) = \begin{cases} (1/\sqrt{n!})\hat{v}(\{x_1, \dots, x_n\}) \\ \quad (x_1, \dots, x_n) \text{ has no coinciding points,} \\ 0 \quad \text{otherwise.} \end{cases} \tag{3.8}$$

We observe that

$$\begin{aligned} \langle f_n, f_n \rangle_n &= \sum_{\underline{x} \in \mathbb{Z}^n} |f_n(x_1, \dots, x_n)|^2 \\ &= \frac{1}{n!} \sum_{\underline{x} \in \mathbb{Z}^n, x_\ell \neq x_m, \text{ all } \ell \neq m} |\hat{v}(\{x_1, \dots, x_n\})|^2 = \sum_{\Lambda \subset \mathbb{Z}, |\Lambda| = n} |\hat{v}(\Lambda)|^2. \end{aligned} \tag{3.9}$$

Hence the map defined by (3.8) is unitary. The set of all $\hat{v}$'s such that the right hand side of (3.9) is finite defines the subspace $P\mathfrak{F}_n$ of $\mathfrak{F}_n$. $P\mathfrak{F}_n$ consists of functions which vanish whenever their argument $(x_1, \dots, x_n)$ has coinciding points. P is a projection operator on $\mathfrak{F}_n$ and P as an operator on $\mathfrak{F}$ is defined through its action on each subspace $\mathfrak{F}_n$.

Now, if $f_{n+1} \in P\mathfrak{F}_{n+1}$, then in the expression (2.31) for A the two last summands vanish. A carries the prefactor $\sqrt{n+1}$ which is balanced by the normalization $1/\sqrt{n!}$ in (3.8). Thus we conclude that, for $\lambda = p$,

$$PHP = H_{\mathrm{AS}} \tag{3.10}$$

on Fock space. Restricting the lattice KPZ Hamiltonian to the subspace $P\mathfrak{F}$ results in the Hamiltonian of the asymmetric simple exclusion process. Note that L_{AS} is the generator of a Markov jump process only for $|p| \leq 1$. On the other hand, H is defined for all λ and thus (3.10) provides an "analytic continuation" of L_{AS}.

It would be most useful to have comparison inequalities between lattice KPZ and ASEP. For the symmetric part of the lattice KPZ Hamiltonian, the quadratic form on $\mathfrak{F}_n$ reads

$$\langle f_n, H_0 f_n \rangle_n = \sum_{\ell=1}^{n} \sum_{x_1, \dots, x_n \in \mathbb{Z}} |f_n(x_1, \dots, x_\ell + 1, \dots, x_n) - f_n(x_1, \dots, x_n)|^2. \tag{3.11}$$

Thus H_0 restricted to $\mathfrak{F}_n$ is the Laplacian on $\mathbb{Z}^n$. On $\mathfrak{F}_n$ the projected operator PH_0P corresponds to the Neumann restriction of the Laplacian (3.11) to the set $\{\underline{x} \in \mathbb{Z}^n | x_\ell \neq x_m$ for all pairs $\ell \neq m\}$. Hence for the symmetric part it holds

$$H_0 \geq PH_0P \tag{3.12}$$

on $\mathfrak{F}$.

On the other hand, there seems to be no such simple relation for the asymmetric part. In [17] the authors approximate the ASEP by a model without hard exclusion, in such a way that its restriction to the range of P agrees with the ASEP. Restricted to the subspace $\bigoplus_{m=0}^{n} \mathfrak{F}_m$, they bound the resolvent of one Hamiltonian in terms of the resolvent of the other Hamiltonian, and vice versa. Their bounds are not uniform in n. Since the approximation in [17] has some similarity to the lattice KPZ Hamiltonian, we expect that their bounds would carry over to H from (2.28).

4 Variance of the Two-Point Function for Large t

Since, by the conservation law,

$$\chi = \sum_{j\in\mathbb{Z}} S(j,0) = \sum_{j\in\mathbb{Z}} S(j,t), \qquad \chi = \frac{1}{2\alpha}, \tag{4.1}$$

and since, by (2.34),

$$\sum_{j\in\mathbb{Z}} j S(j,t) = 0, \tag{4.2}$$

it is natural to introduce the normalized variance

$$\mathrm{Var}\,(t) = \chi^{-1} \sum_{j\in\mathbb{Z}} j^2 S(j,t). \tag{4.3}$$

We first represent $S(j,t)$ through the transition probability e^{Lt}, where L is the generator L_N from (2.5) with the sum over all $j \in \mathbb{Z}$. Then

$$S(j,t) = \mathbb{E}(u_0(0)u_j(t)) = \langle u_0 \mathrm{e}^{Lt} u_j\rangle. \tag{4.4}$$

The expectation $\mathbb{E}$ has been defined below (2.32) and $\langle\cdot\rangle$ refers to the average over the initial data, i.e. over independent Gaussians with mean 0 and variance $1/2\alpha$. u_0 and $\mathrm{e}^{Lt}u_j$ are functions on configuration space. The conservation law is used again to partially integrate twice,

$$\begin{aligned}
\chi\,\mathrm{Var}\,(t) &= \sum_{j\in\mathbb{Z}} j^2 \langle u_0 \mathrm{e}^{Lt} u_j\rangle \\
&= \sum_{j\in\mathbb{Z}} j^2 \left(\langle u_0 u_j\rangle + \int_0^t \mathrm{d}s\, \langle u_0 \mathrm{e}^{Ls} L u_j\rangle \right) \\
&= \int_0^t \mathrm{d}s \sum_{j\in\mathbb{Z}} j^2 \langle u_0 \mathrm{e}^{Ls}(w_j - w_{j-1})\rangle \\
&= \int_0^t \mathrm{d}s \sum_{j\in\mathbb{Z}} (-2j-1) \left(\langle u_0 w_j\rangle + \int_0^s \mathrm{d}s' \langle (L^* u_0) \mathrm{e}^{Ls'} w_j\rangle \right) \\
&= 2t\nu_0 \langle u_0^2\rangle + 2\int_0^t \mathrm{d}s \int_0^s \mathrm{d}s' \sum_{j\in\mathbb{Z}} \big(\langle (w_0 - \langle w_0\rangle) \mathrm{e}^{Ls'} (w_j - \langle w_j\rangle)\rangle \big).
\end{aligned} \tag{4.5}$$

It is convenient to switch to Laplace transform. Then

$$\begin{aligned}
\chi\,\hat{\mathrm{V}}\mathrm{ar}(\zeta) &= \int_0^\infty \mathrm{d}t \mathrm{e}^{-\zeta\nu_0 t} \chi\,\mathrm{Var}\,(t) \\
&= 2\zeta^{-2}\nu_0^{-1}\langle u_0^2\rangle + 2(\nu_0\zeta)^{-2} \sum_{j\in\mathbb{Z}} \left\langle (w_0 - \langle w_0\rangle) \frac{1}{\nu_0\zeta - L} (w_j - \langle w_j\rangle) \right\rangle
\end{aligned} \tag{4.6}$$

for $\zeta > 0$.

On the other hand, the KPZ scaling theory asserts that

$$S(j,t) \cong \chi(2\lambda_0^2\chi t^2)^{-1/3} f_{\mathrm{KPZ}}\big((2\lambda_0^2\chi t^2)^{-1/3} j\big) \tag{4.7}$$

with χ defined in (4.1), the renormalized coupling constant $j''(0) = \lambda_0$, and the universal stationary KPZ scaling function f_{KPZ}. The validity of (4.7) has been proved for the stationary PNG model [4] and for the stationary TASEP [5]. We refer to these papers for the definition of f_{KPZ}. In particular,

$$\mathrm{Var}\,(t) = (2\lambda_0^2\chi t^2)^{2/3}\langle x^2\rangle_{\mathrm{KPZ}} \tag{4.8}$$

for $t \to \infty$ with $\langle x^2\rangle_{\mathrm{KPZ}} = \int dx x^2 f_{\mathrm{KPZ}}(x) = 0.510523\ldots$ a model independent number. Hence

$$\hat{\mathrm{V}}\mathrm{ar}\,(\zeta) = (2\lambda_0^2\chi)^{2/3}\Gamma(7/3)(\nu_0\zeta)^{-7/3}\langle x^2\rangle_{\mathrm{KPZ}} \tag{4.9}$$

for $\zeta \to 0$. Comparing (4.9) and (4.6) yields the prediction for the small ζ behavior of the resolvent in (4.6). But before we have to reexpress this resolvent in Fock space.

Let us define the total momentum operator, P_{tot}, as

$$P_{\mathrm{tot}} = 2\pi \int_{\mathbb{T}} dk k a(k)^* a(k). \tag{4.10}$$

Since H is translation invariant,

$$[H, P_{\mathrm{tot}}] = 0, \tag{4.11}$$

there exists the direct integral decompositions corresponding to P_{tot} as

$$\mathfrak{F} = \int_{\mathbb{T}}^{\oplus} dk \mathfrak{F}(k) \tag{4.12}$$

and

$$H = \int_{\mathbb{T}}^{\oplus} dk H(k). \tag{4.13}$$

We will need only the fiber at $k = 0$. To construct $\mathfrak{F}(0)$ we consider $\hat{f} \in \mathfrak{F}$, in the momentum representation, such that $\hat{f}_0 = 0$ and $\hat{f}_n$ are continuous on $\mathbb{T}^n$. Then $\hat{f}^0 \in \mathfrak{F}(0)$ is defined by

$$\hat{f}_n^0(k_1,\ldots,k_n) = \hat{f}_n(k_1,\ldots,k_n)\delta(k_1+\cdots+k_n), \quad n = 1,2,\ldots. \tag{4.14}$$

The scalar product is given by

$$\langle \hat{f}_n^0, \hat{f}_n^0\rangle_n^0 = \int_{\mathbb{T}^n} dk_1\ldots dk_n \delta(k_1+\cdots+k_n)|\hat{f}_n(k_1,\ldots,k_n)|^2, \tag{4.15}$$

$$\langle \hat{f}^0, \hat{f}^0\rangle^0 = \sum_{n=1}^{\infty}\langle \hat{f}_n^0, \hat{f}_n^0\rangle_n^0. \tag{4.16}$$

$\mathfrak{F}(0)$ is the completion with respect to this norm.

The free part on $\mathfrak{F}_n$ is multiplication by

$$\Omega_n(k_1,\ldots,k_n) = \sum_{\ell=1}^{n}\omega(k_\ell). \tag{4.17}$$

Hence

$$(H_0\hat{f}_n^0)(k_1,\dots,k_n)=\Omega_n(k_1,\dots,k_n)\hat{f}_n(k_1,\dots,k_n)\delta(k_1+\cdots+k_n) \tag{4.18}$$

and $H_0\hat{f}^0\in\mathfrak{F}(0)$. Correspondingly, using (2.40), the cubic operator A on $\mathfrak{F}$ acts as

$$\begin{aligned}(A\hat{f}_{n+1})(k_1,\dots,k_n)=-2i\sqrt{n+1}\Bigg(\sum_{\ell=1}^{n}\int_{\mathbb{T}}\mathrm{d}k_{n+1}\big(\sin(2\pi k_\ell)+\sin(2\pi k_{n+1})\big)\\ \times\hat{f}_{n+1}(k_1,\dots,\hat{k}_\ell,\dots,k_{n+1},k_\ell-k_{n+1})\Bigg)\end{aligned} \tag{4.19}$$

with a similar expression for A^*, see below. Clearly, if $\hat{f}_{n+1}$ contains the delta function $\delta(k_1+\cdots+k_{n+1})$, then $A\hat{f}_{n+1}$ is proportional to $\delta(k_1+\cdots+k_n)$ and hence $A\hat{f}_{n+1}^0\in\mathfrak{F}_n(0)$. Note that $H\restriction\mathfrak{F}_1(0)=0$ and $H^*\restriction\mathfrak{F}_1(0)=0$.

The current function at $j=0$ with its average subtracted equals $w_0-\langle w_0\rangle=\nu_0(u_1-u_0)+(\lambda_0/6)(u_0^2+u_0u_1+u_1^2-\alpha^{-1})$. To construct the corresponding vector $f^w\in\mathfrak{F}$, we use

$$u_j\psi_0=(2\alpha)^{-1/2}a_j^*\psi_0,\qquad \big(u_j^2-(2\alpha)^{-1}\big)\psi_0=(2\alpha)^{-1}a_j^{*2}\psi_0, \tag{4.20}$$

and the fact that the ground state ψ_0 is mapped to the Fock vacuum. By using (2.27) one obtains $f_n^w=0$ for $n=0$, $n\geq 3$, and

$$\begin{aligned}f_1^w(x_1)&=\nu_0(2\alpha)^{-1/2}\big(\delta(x_1-1)-\delta(x_1)\big),\\ f_2^w(x_1,x_2)&=(\lambda_0/6)(2\sqrt{2}\alpha)^{-1}\big(2\delta(x_1)\delta(x_2)+2\delta(x_1-1)\delta(x_2-1)\\ &\quad+\delta(x_1)\delta(x_2-1)+\delta(x_1-1)\delta(x_2)\big),\end{aligned} \tag{4.21}$$

which in Fourier space reads

$$\begin{aligned}\hat{f}_1^w(k_1)&=\nu_0(2\alpha)^{-1/2}(\mathrm{e}^{-i2\pi k_1}-1),\\ \hat{f}_2^w(k_1,k_2)&=(\lambda_0/6)(2\sqrt{2}\alpha)^{-1}\big(2+2\mathrm{e}^{-i2\pi(k_1+k_2)}+\mathrm{e}^{-i2\pi k_1}+\mathrm{e}^{-i2\pi k_2}\big).\end{aligned} \tag{4.22}$$

$\hat{f}^{w0}\in\mathfrak{F}(0)$ is then given by

$$\begin{aligned}&\hat{f}^{w0}=\lambda_0(2\sqrt{2}\alpha)^{-1}\hat{g}^0,\quad \hat{g}^0=(0,0,\hat{w}^0,0,0,\dots),\\ &\hat{w}^0(k_1,k_2)=\hat{w}(k_1)\delta(k_1+k_2),\quad \hat{w}(k_1)=\frac{1}{3}(2+\cos(2\pi k_1)),\end{aligned} \tag{4.23}$$

enforcing the normalization $\hat{w}(0)=1$.

With this input the resolvent from (4.6) reads

$$\sum_{j\in\mathbb{Z}}\langle(w_0-\langle w_0\rangle)\frac{1}{\nu_0\zeta-L}(w_j-\langle w_j\rangle)\rangle=\lambda_0^2(2\sqrt{2}\alpha)^{-2}\nu_0^{-1}\langle\hat{g}^0,(\zeta+H)^{-1}\hat{g}^0\rangle^0. \tag{4.24}$$

Comparing (4.9) and (4.6) together with (4.24) one arrives at the following prediction from the KPZ scaling theory,

$$\langle\hat{g}^0,(\zeta+H)^{-1}\hat{g}^0\rangle^0=3^{-2/3}\Gamma(7/3)\langle x^2\rangle_{\mathrm{KPZ}}(\lambda^2\zeta)^{-1/3} \tag{4.25}$$

for small ζ.

5 Bounds and Relaxation Time Approximation

While at present we have no techniques to establish (4.25), following the methods in [17] one can study upper and lower bounds for the matrix element $\langle \hat{g}^0, (\zeta + H)^{-1}\hat{g}^0\rangle^0$. Restricting H to $\bigoplus_{m=1}^{n}\mathfrak{F}_m(0)$ yields bounds depending on n, which alternate as upper and lower bounds and which converge to a limit as $n \to \infty$. More precisely, we define recursively

$$\begin{aligned} T_2 &= (\zeta + H_0)^{-1}, \\ T_n &= (\zeta + H_0 + \lambda^2 A T_{n-1} A^*)^{-1}, \end{aligned} \tag{5.1}$$

and set

$$b_n(\zeta) = \langle \hat{g}^0, T_n \hat{g}^0\rangle^0. \tag{5.2}$$

E.g.,

$$\begin{aligned} b_3(\zeta) &= \langle \hat{g}^0, \{\zeta + H_0 + \lambda^2 A(\zeta + H_0)^{-1}A^*\}^{-1}\hat{g}^0\rangle^0, \\ b_4(\zeta) &= \langle \hat{g}^0, \{\zeta + H_0 + \lambda^2 A(\zeta + H_0 + \lambda^2 A(\zeta + H_0)^{-1}A^*)^{-1}A^*\}^{-1}\hat{g}^0\rangle^0. \end{aligned} \tag{5.3}$$

Then

$$b_3(\zeta) \le b_5(\zeta) \le \cdots \le b_4(\zeta) \le b_2(\zeta), \tag{5.4}$$

and

$$\langle \hat{g}^0, (\zeta + H)^{-1}\hat{g}^0\rangle^0 = \lim_{n\to\infty} b_n(\zeta). \tag{5.5}$$

The vector $\hat{g}^0$ has $\hat{g}^0_2 = \hat{w}^0$ as the only nonvanishing entry. H_0 preserves the particle number, while A decreases it by one and A^* increases it by one. Hence it will be convenient to introduce a notation displaying these special features. Let P_n be the projection onto $\mathfrak{F}_n$. We set

$$P_n H_0 P_n = H_{0,n} \tag{5.6}$$

and

$$P_n A P_{n+1} = A_{n,n+1}, \qquad P_{n+1} A^* P_n = A_{n+1,n}. \tag{5.7}$$

For a given n, we define recursively

$$U^{(n)}_{n-1} = \lambda^2 A_{n-1,n}(\zeta + H_{0,n})^{-1}A_{n,n-1}, \tag{5.8}$$

$$U^{(n)}_m = \lambda^2 A_{m,m+1}(\zeta + H_{0,m+1} + U^{(n)}_{m+1})^{-1}A_{m+1,m}, \quad 2 \le m \le n-2. \tag{5.9}$$

$U^{(n)}_m$ acts on $\mathfrak{F}_m$ and leaves $\mathfrak{F}_m(0)$ invariant. Our bound $b_n(\zeta)$ can then be written as

$$b_n(\zeta) = \langle \hat{w}^0, (\zeta + H_{0,2} + U^{(n)}_2)^{-1}\hat{w}^0\rangle^0_2. \tag{5.10}$$

The next task is to work out more concretely the operator $A_{n,n+1}(\zeta + H_{0,n+1})^{-1}\ A_{n+1,n}$ appearing in (5.8). We use duality to compute A^*,

$$\begin{aligned}\langle f_n, Af_{n+1}\rangle_n &= \int_{\mathbb{T}^n} \mathrm{d}\underline{k}\, f_n(k_1,\dots,k_n)^* Af_{n+1}(k_1,\dots,k_n)\\ &= -2i\sqrt{n+1}\sum_{\ell=1}^{n}\int_{\mathbb{T}^{n+1}} \mathrm{d}\underline{k}\mathrm{d}k_{n+1}\big(\sin(2\pi k_\ell)+\sin(2\pi k_{n+1})\big) f_n(k_1,\dots,k_n)^*\\ &\quad\times f_{n+1}(k_1,\dots,k_\ell-k_{n+1},\dots,k_{n+1})\\ &= \int_{\mathbb{T}^{n+1}} \mathrm{d}\underline{k}\mathrm{d}k_{n+1} f_{n+1}(k_1,\dots,k_{n+1})\Bigg(2i\sqrt{n+1}\sum_{\ell=1}^{n}\big(\sin(2\pi(k_\ell+k_{n+1}))\\ &\quad+\sin(2\pi k_{n+1})\big) f_n(k_1,\dots,k_\ell+k_{n+1},\dots,k_n)\Bigg)^*. \end{aligned} \tag{5.11}$$

Upon symmetrization, one arrives at

$$\begin{aligned}A^* f_n(k_1,\dots,k_{n+1}) &= 2i\frac{1}{\sqrt{n+1}}\sum_{1\le j<\ell\le n+1}\big(2\sin(2\pi(k_j+k_\ell))+\sin(2\pi k_j)\\ &\quad+\sin(2\pi k_\ell)\big) f_n(k_1,\dots,\hat{k}_j,\hat{k}_\ell,\dots,k_j+k_\ell).\end{aligned} \tag{5.12}$$

To compute $U_n^{(n+1)}$ we denote the shift $k_i \rightsquigarrow k_i - k_{n+1}$ by τ_i. Then

$$\begin{aligned}U_n^{(n+1)} f_n(k_1,\dots,k_n) &= \lambda^2 (A_{n,n+1}(\zeta+H_{0,n+1})^{-1}A_{n+1,n} f_n)(k_1,\dots,k_n)\\ &= 4\lambda^2\sum_{i=1}^{n}\sum_{1\le j<\ell\le n+1}\int_{\mathbb{T}}\mathrm{d}k_{n+1}\big(\sin(2\pi(k_i-k_{n+1}))+\sin(2\pi k_{n+1})\big)\\ &\quad\times(\zeta+\Omega_{n+1}(k_1,\dots,k_\ell-k_{n+1},\dots,k_{n+1}))^{-1}\tau_i\big(\sin(2\pi k_j)\\ &\quad+\sin(2\pi k_\ell)+2\sin(2\pi(k_j+k_\ell))\big)\tau_i f_n(k_1,\dots,\hat{k}_j,\hat{k}_\ell,\dots,k_j+k_\ell).\end{aligned} \tag{5.13}$$

Clearly, $U_n^{(n+1)} = V_n^{(n+1)} + R_n^{(n+1)}$, where $V_n^{(n+1)}$ is a multiplication operator and $R_n^{(n+1)}$ an integral operator. In the sum in (5.13), $V_n^{(n+1)}$ corresponds to the terms $i=j$ and $\ell=n+1$. Hence

$$\begin{aligned}V_n^{(n+1)}(k_1,\dots,k_n) &= 2\lambda^2\sum_{\ell=1}^{n}\int_{\mathbb{T}}\mathrm{d}k_{n+1}\big(\sin(2\pi(k_\ell-k_{n+1}))+\sin(2\pi k_{n+1})+2\sin(2\pi k_\ell)\big)^2\\ &\quad\times(\zeta+\Omega_{n+1}(k_1,\dots,k_\ell-k_{n+1},\dots,k_{n+1}))^{-1}.\end{aligned} \tag{5.14}$$

Considering the special cases $n=2,3$ one arrives at the following bounds.

Theorem 5.1 *In the limit $\zeta\to 0$, the following bounds are valid,*

$$\lambda^{-1}2^{-5/4}3^{-3/2}\zeta^{-1/4} \le \langle \hat{g}^0, (\zeta+H)^{-1}\hat{g}^0\rangle^0 \le 2^{-3/2}\zeta^{-1/2}. \tag{5.15}$$

While the lower bound is not sharp, it establishes that $S(j,t)$ must spread superdiffusively.

Proof $n=2$: It holds

$$\begin{aligned}\langle \hat{g}^0, \big(\zeta+H\big)^{-1}\hat{g}^0\rangle^0 &\le \langle \hat{w}^0, \big(\zeta+H_{0,2}\big)^{-1}\hat{w}^0\rangle_2^0 \\ &= \int_{\mathbb{T}} \mathrm{d}k_1 \Big(\frac{1}{3}(2+\cos(2\pi k_1))\Big)^2 \big(\zeta+2\omega(k_1)\big)^{-1} \\ &\cong 2^{-3/2}\zeta^{-1/2} \end{aligned} \tag{5.16}$$

for $\zeta\to 0$.

$n=3$: It holds

$$\langle \hat{g}^0, (\zeta+H)^{-1}\hat{g}^0\rangle^0 \ge \langle \hat{w}^0, [\zeta+H_{0,2}+\lambda^2 A_{2,3}(\zeta+H_{0,3})^{-1}A_{3,2}]^{-1}\hat{w}^0\rangle_2^0. \tag{5.17}$$

For the quadratic form of $\lambda^2 A_{2,3}(\zeta+H_{0,3})^{-1}A_{3,2}$ one obtains, with $\hat{f}_2\in\mathfrak{F}_2$,

$$\begin{aligned}&\lambda^2\langle \hat{f}_2, A_{2,3}(\zeta+H_{0,3})^{-1}A_{3,2}\hat{f}_2\rangle_2 \\ &= (4\lambda^2/3)\int_{\mathbb{T}^3}\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3\big(\zeta+\Omega_3(k_1,k_2,k_3)\big)^{-1} \\ &\quad\times\Big(\big(2\sin(2\pi(k_1+k_2))+\sin(2\pi k_1)+\sin(2\pi k_2)\big)\hat{f}_2(k_1+k_2,k_3) \\ &\quad+\big(2\sin(2\pi(k_1+k_3))+\sin(2\pi k_1)+\sin(2\pi k_3)\big)\hat{f}_2(k_1+k_3,k_2) \\ &\quad+\big(2\sin(2\pi(k_2+k_3))+\sin(2\pi k_2)+\sin(2\pi k_3)\big)\hat{f}_2(k_2+k_3,k_1)\Big)^2 \\ &\le 3\langle \hat{f}_2, V_2^{(3)}\hat{f}_2\rangle_2, \end{aligned} \tag{5.18}$$

where Schwarz inequality is used in the last step. By taking limits, the inequality holds also for the fiber $k=0$. Hence

$$\langle \hat{g}^0, (\zeta+H)^{-1}\hat{g}^0\rangle^0 \ge \langle \hat{w}^0, (\zeta+H_{0,2}+3V_2^{(3)})^{-1}\hat{w}^0\rangle_2^0. \tag{5.19}$$

For small ζ the dominant part of the integral comes from k_1 close to 0 and one obtains

$$\begin{aligned}\langle \hat{w}^0, (\zeta+H_{0,2}+3V_2^3)^{-1}\hat{w}^0\rangle_2^0 &\cong \int_{\mathbb{T}}\mathrm{d}k_1(\zeta+(2\pi k_1)^2\lambda^2 2^{1/2}3^3\zeta^{-1/2})^{-1} \\ &= 2^{-1}(\lambda^2 2^{1/2}3^3)^{-1/2}\zeta^{-1/4}. \end{aligned} \tag{5.20}$$

□

It is of interest to understand whether Theorem 5.1 could be improved to yield the KPZ exponent $1/3$ as $n\to\infty$. Unfortunately, already for $n=4$, one would need a lower bound on $A_{3,4}(\zeta+H_{0,4})A_{4,3}$ by a multiplication operator, compare with (5.18). Such a bound does not seem to be available. To make, nevertheless, some progress we observe the splitting

$$U_{n-1}^{(n)} = V_{n-1}^{(n)} + R_{n-1}^{(n)} \tag{5.21}$$

in (5.13), where $V_{n-1}^{(n)}$ is a multiplication operator and the remainder $R_{n-1}^{(n)}$ is an integral operator involving either a single particle or a pair of particles. Therefore one expects that the limit $\zeta\to 0$ will be dominated by the potential $V_{n-1}^{(n)}$. In solid state physics, in the context of transport equations, this type of approximation is called the relaxation time approximation, see [19], e.g.. Of course it is uncontrolled, at least at present.

With this approximation in the expression for $U^{(n)}_{n-2}$ one substitutes $V^{(n)}_{n-1}$ for $U^{(n)}_{n-1}$. Again for $U^{(n)}_{n-2}$ one neglects the integral operator $R^{(n)}_{n-2}$. Iterating this procedure yields the recursion relation for the potentials $V^{(n)}_m$,

$$\begin{aligned}
&V^{(n)}_n = 0,\\
&V^{(n)}_m(k_1,\dots,k_m) = 2\lambda^2 \sum_{\ell=1}^{m} \int_{\mathbb{T}} \mathrm{d}k_{m+1} \big(2\sin(2\pi k_\ell) + \sin(2\pi(k_\ell - k_{m+1})) + \sin(2\pi k_{m+1})\big)^2\\
&\qquad\qquad \times \big(\zeta + \Omega_m(k_1,\dots,k_\ell - k_{m+1},\dots,k_{m+1})\\
&\qquad\qquad + V^{(n)}_{m+1}(k_1,\dots,k_\ell - k_{m+1},\dots,k_{m+1})\big)^{-1}, \quad 2 \le m \le n-1.
\end{aligned} \tag{5.22}$$

Denoting $b_n(\zeta)$ within this approximation by $d_n(\zeta)$, one arrives at

$$d_n(\zeta) = \int_{\mathbb{T}} \mathrm{d}k \frac{|\hat{w}(k)|^2}{\zeta + 2\omega(k) + V_2(k,-k)}. \tag{5.23}$$

Here $\hat{w}(k)$ is defined in (4.23), $\hat{w}(0) = 0$.

For small ζ the dominant contribution to the integral (5.22) comes from k_{m+1} close to 0. Let us start with $m = n-1$. Then

$$\begin{aligned}
V^{(n)}_{n-1}(k_1,\dots,k_{n-1}) &\cong (2\lambda^2)9\Omega_{n-1}(k_1,\dots,k_{n-1}) \int_{\mathbb{T}} \mathrm{d}k_n (\zeta + 2\omega(k_n))^{-1}\\
&\cong 9\lambda^2 (2\zeta)^{-1/2} \Omega_{n-1}(k_1,\dots,k_{n-1}).
\end{aligned} \tag{5.24}$$

The next iteration reads

$$\begin{aligned}
V^{(n)}_{n-2}(k_1,\dots,k_{n-2}) &\cong (2\lambda^2)9\Omega_{n-2}(k_1,\dots,k_{n-2}) \int_{\mathbb{T}} \mathrm{d}k_{n-1} (\zeta + (9\lambda^2)2\omega(k_{n-1})(2\zeta)^{-1/2})^{-1}\\
&\cong (9\lambda^2)^{1/2} (2\zeta)^{-1/4} \Omega_{n-2}(k_1,\dots,k_{n-2}),
\end{aligned} \tag{5.25}$$

and, in general,

$$V^{(n)}_m(k_1,\dots,k_m) \cong (9\lambda^2)^{1-\alpha_{n-m}} (2\zeta)^{-\alpha_{n-m+1}} \Omega_m(k_1,\dots,k_m), \quad 2 \le m \le n-1, \tag{5.26}$$

where the exponents α_j are defined recursively through

$$\alpha_{j+1} = \frac{1}{2}(1 - \alpha_j), \qquad \alpha_1 = 0. \tag{5.27}$$

Substituting (5.26) in (5.23) and using $\hat{w}(0) = 1$ yields for small ζ

$$d_n(\zeta) = 2^{-1} (9\lambda^2)^{-\alpha_{n-1}} (2\zeta)^{-\alpha_n}. \tag{5.28}$$

The solution to (5.27) reads

$$\alpha_n = \frac{1}{3}\big(1 - (-2)^{-(n-1)}\big), \tag{5.29}$$

which for $n \to \infty$ converges to $\alpha_\infty = 1/3$. Therefore $\lim_{n\to\infty} d_n(\zeta) = d_\infty(\zeta)$ and

$$d_\infty(\zeta) = 2^{-4/3} 3^{-2/3} (\lambda^2 \zeta)^{-1/3}, \tag{5.30}$$

which should be compared with (4.25). Remarkably enough, the relaxation time approximation yields the KPZ exponent 1/3. The prefactor of $(\lambda^2\zeta)^{-1/3}$ equals 0.292 in (4.25) while it is 0.191 in (5.30). Thus the relaxation time approximation gives a prefactor which is approximately 2/3 off the true value. We take this as an indication that one needs more powerful methods to obtain the universal scaling form for the two-point function.

6 Continuum Limit

As remarked in the Introduction, the physically meaningful continuum limit of (1.5) must be such as to preserve the scaling (4.7) of the two-point function. Here we want to amplify this point and start on the unit lattice. To avoid the various constants, we make the specific choice $\nu_0 = 1/2$, $D_0 = 1$. Let us first consider the linear case, $\lambda_0 = 0$. Then on the microscopic scale

$$\frac{d}{dt} u_j(t) = \frac{1}{2}\big(u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)\big) + \xi_j(t) - \xi_{j-1}(t). \tag{6.1}$$

For the continuum limit we average over a smooth test function g, $g : \mathbb{R} \to \mathbb{R}$, varying on the scale δ^{-1}, $\delta \ll 1$, and take long times $\delta^{-2}t$ with $t = \mathcal{O}(1)$. Then

$$\lim_{\delta\to 0} \delta \sum_{j\in\mathbb{Z}} g(\delta j)\delta^{-1/2} u_j(\delta^{-2}t) = \int \mathrm{d}x\, g(x)\phi(x,t), \tag{6.2}$$

where ϕ is a mean zero Gaussian field with covariance

$$\langle \phi(x,t)\phi(x',t')\rangle = (2\pi|t-t'|)^{-1/2} \exp\big[-(x-x')^2/2|t-t'|\big]. \tag{6.3}$$

Equivalently one can view u_j on the lattice with spacing δ by defining

$$v^\delta(x,t) = \delta^{-1/2} u_{\lfloor x/\delta \rfloor}(\delta^{-2}t) \tag{6.4}$$

with $\lfloor\cdot\rfloor$ denoting integer part and $x \in \mathbb{R}$. Then, integrating against smooth test functions, the continuum limit reads

$$\lim_{\delta\to 0} v^\delta(x,t) = \phi(x,t). \tag{6.5}$$

By (6.1) v^δ satisfies

$$\begin{aligned} \frac{d}{dt} v^\delta(x,t) &= \delta^{-2}\frac{1}{2}\big(v^\delta(x+\delta,t) - 2v^\delta(x,t) + v^\delta(x-\delta,t)\big) \\ &\quad + \delta^{-1/2}\delta^{-1}\big(\xi_{\lfloor x/\delta\rfloor}(t) - \xi_{\lfloor (x-\delta)/\delta\rfloor}(t)\big). \end{aligned} \tag{6.6}$$

Comparing with (1.5), we observe that the bare coefficients are not rescaled. Taking the limit $\delta \to 0$ in (6.6) yields

$$\partial_t \phi(x,t) = \frac{1}{2}\partial_x^2 \phi(x,t) + \partial_x \xi(x,t) \tag{6.7}$$

with white noise initial data, in agreement with (6.3), (6.5).

Next we include the nonlinearity with $\lambda_0 = 1$. Then

$$\frac{d}{dt}u_j(t) = \tilde{w}_j(t) - \tilde{w}_{j-1}(t) + \frac{1}{2}\big(u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)\big) + \xi_j(t) - \xi_{j-1}(t) \tag{6.8}$$

and the stationary two-point function should scale as

$$\lim_{\delta\to 0} \delta S([\delta^{-1}x], \delta^{-3/2}t) = t^{-2/3} f_{\mathrm{KPZ}}(t^{-2/3}x). \tag{6.9}$$

In particular, the correct time scale for the continuum limit is $\delta^{-3/2}t$, $t = \mathcal{O}(1)$. In analogy to (6.5) one introduces the macroscopic field $v^\delta(x,t)$ by

$$v^\delta(x,t) = \delta^{-1/2} u_{\lfloor x/\delta \rfloor}(\delta^{-3/2}t). \tag{6.10}$$

According to (6.8), $v^\delta(x,t)$ is governed by the evolution

$$\begin{aligned}\frac{d}{dt}v^\delta(x,t) &= \delta^{-1}\big(\tilde{w}^\delta(x,t) - \tilde{w}^\delta(x-\delta,t)\big)\\ &\quad + \delta^{1/2}\delta^{-2}\frac{1}{2}\big(v^\delta(x+\delta,t) - 2v^\delta(x,t) + v^\delta(x-\delta,t)\big)\\ &\quad + \delta^{1/4}\delta^{-1/2}\delta^{-1}\big(\xi_{\lfloor x/\delta\rfloor}(t) - \xi_{\lfloor (x-\delta)/\delta\rfloor}(t)\big).\end{aligned} \tag{6.11}$$

Comparing with (1.5) we conclude that the nonlinearity is left invariant, thus λ_{b} remains fixed, while the linear part is scaled down by $\sqrt{\delta}$. Of course, it is natural to conjecture that, as in the linear case, $v^\delta(x,t)$ has a limit as $\delta \to 0$. To identify the limit one could also use other, better understood models, like the TASEP. Some properties are known [5, 18], but the full limit of $v^\delta(x,t)$ still has to be identified.

There is one particular case for which analytical results are available [20, 21]. In fact, it is the problem Burgers wanted to solve [22]. In (1.2) one sets $D_{\mathrm{b}} = 0$ and studies the decay of the solution for Gaussian white noise initial data, $\langle u(x,0)u(x',0)\rangle = (1/8)\delta(x-x')$. We set $\lambda_{\mathrm{b}} = 1$. The solution to (1.2) is well defined in the limit $\nu_{\mathrm{b}} = 0$. With this meaning, we set $\nu_{\mathrm{b}} = 0$. Then the solution $u(x,t)$ is statistically self-similar, in the sense that

$$u(x,t) = t^{-1/3}u(t^{-2/3}x, 1) \tag{6.12}$$

in distribution. $x \to u(x,1)$ is a stationary Markov process with the generator

$$L_{\mathrm{T}} f(u) = \frac{d}{du}f(u) + \int_{-\infty}^{u} \mathrm{d}u' R(u,u')\big(f(u') - f(u)\big) \tag{6.13}$$

for functions $f : \mathbb{R} \to \mathbb{R}$. The second term corresponds to a Markov jump process. Thus $u(x,1) = a + x$ locally, interrupted by downward jumps from u to u' with rate $R(u,u')$. The rate function is computed explicitly and given by

$$R(u,u') = (u-u')\frac{J(u')}{J(u)} I(u-u'), \tag{6.14}$$

where I and J are given by their Fourier and Laplace transforms in terms of the Airy function Ai,

$$J(u) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dz \frac{\exp(uz)}{2^{1/3}\mathrm{Ai}(2^{-1/3}z)}, \tag{6.15}$$

$$2I(u) = (2\pi u^3)^{-1/2} + \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dz \exp(uz)\Big(\frac{2^{2/3}\mathrm{Ai}'(2^{-1/3}z)}{\mathrm{Ai}(2^{-1/3}z)} + (2z)^{1/2}\Big). \tag{6.16}$$

Plots of the stationary distribution for L_T, and other quantities, can be found in [20].

References

1. Johansson, K.: Random matrices and determinantal processes. In: Bovier, A., et al. (eds.) Math. Stat. Phys., Session LXXXIII: Lecture Notes of the Les Houches Summer School 2005, pp. 1–56. Elsevier Science, Amsterdam (2006)
2. Spohn, H.: Exact solutions for KPZ-type growth processes, random matrices, and equilibrium shapes of crystals. Physica A **369**, 71–99 (2006)
3. Sasamoto, T.: Fluctuations of the one-dimensional asymmetric exclusion process using random matrix techniques. J. Stat. Mech. P07007 (2007)
4. Prähofer, M., Spohn, H.: Exact scaling function for one-dimensional stationary KPZ growth. J. Stat. Phys. **115**, 255–279 (2004)
5. Ferrari, P.L., Spohn, H.: Scaling limit for the space-time covariance of the stationary totally asymmetric simple exclusion process. Commun. Math. Phys. **265**, 1–44 (2006)
6. Kardar, M., Parisi, G., Zhang, Y.Z.: Dynamic scaling of growing interfaces. Phys. Rev. Lett. **56**, 889–892 (1986)
7. Katzav, E., Schwartz, M.: Numerical evidence for stretched exponential relaxations in the Kardar-Parisi-Zhang equation. Phys. Rev. E **69**, 052603 (2004)
8. Bertini, L., Giacomin, G.: Stochastic Burgers and KPZ equations from particle systems. Commun. Math. Phys. **183**, 571–607 (1997)
9. Balazs, M., Quastel, J., Seppalainen, T.: Scaling exponent for the Cole-Hopf solution of KPZ/stochastic Burgers. arXiv:0909.4816
10. Bernardin, C.: Superdiffusivity of asymmetric energy model in dimension one and two. J. Math. Phys. **49**, 103301 (2008)
11. Miyao, T.: Private communication (2009)
12. Gotoh, T., Kraichnan, R.H.: Burgers turbulence with large scale forcing. Phys. Fluids A **10**, 2859–2866 (1998)
13. E, W., Khanin, K., Mazel, A., Sinai, Y.: Invariant measures for Burgers equation with stochastic forcing. Ann. Math. **151**, 877–960 (2000)
14. Krug, J., Spohn, H.: Kinetic roughening of growing surfaces. In: Godrèche, C. (ed.) Solids Far from Equilibrium, pp. 412–525. Cambridge University Press, Cambridge (1991)
15. Lam, C.-H., Shin, F.G.: Improved discretization of the Kardar-Parisi-Zhang equation. Phys. Rev. E **58**, 5592–5595 (1998)
16. Schweber, S.: An Introduction to Relativistic Quantum Field Theory. Harper & Row, New York (1966)
17. Landim, C., Quastel, J., Salmhofer, M., Yau, H.-T.: Superdiffusivity of asymmetric exclusion process in dimensions one and two. Commun. Math. Phys. **244**, 455–481 (2004)
18. Baik, J., Ferrari, P., Péché, S.: Limit process for the TASEP near a characteristic line. arXiv:0907.0226
19. Ziman, J.M.: Electrons and Phonons: The Theory of Transport Phenomena in Solids. Oxford University Press, New York (1987)
20. Frachebourg, I., Martin, Ph.A.: Exact statistical properties of the Burgers equation. J. Fluid Mech. **417**, 323–349 (2000)
21. Chabanol, M.-L., Duchon, J.: Markovian solutions of inviscid Burgers equation. J. Stat. Phys. **114**, 525–534 (2004)
22. Burgers, J.M.: The Nonlinear Diffusion Equation. Reidel, Dordrecht (1974)